Exit problem of McKean-Vlasov diffusions in double-wells landscape

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Abstract

We consider a diffusion in which the own law of the process appears in the drift, that is a non-linearity in the sense of McKean. This kind of diffusion is obtained as the hydrodynamical limit of a mean-field system of interacting particles. The question that we deal with is the exit-time of such a diffusion when it evolves in a double-wells landscape. This has already been solved for the convex case but the previous methods relie completely on the convexity of the external force. Here, we provide a Kramers'type law for self-stabilizing process directed by a non-convex confining potential.

Key words and phrases: Self-stabilizing diffusion ; Exit-time ; Large deviations ; Coupling method ; Granular media equation

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1 Introduction

1.1 Aim of the article

The aim of the current work is to solve the exit-problem of the McKean-Vlasov diffusion without convexity assumptions either on the confining potential nor on the interacting one. More precisely, we give a Kramers'type law on the exit-time when the diffusion coefficient is asymptotically small.

The convex case has already been solved by two different methods which can not be extended to the non-convex case.

A recent paper ([Tug14]) solves the non-convex case but under strong hypotheses. These hypotheses seem to be technical but not of crucial interest.

We present the result in the simplest case, that is the one of the linear interaction. However, the result can be extended to a more general setting ; included the case in which the non-linearity of the drift is not of McKean type.

1.2 Model

Let X_0 be any random variable. We are interested in a diffusion $X = (X_t)_{t\geq 0}$ satisfying the following equation:

$$X_t = X_0 + \int_0^t U(X_s)ds - \int_0^t \int_{\mathbb{R}^d} \Phi\left(X_s - x\right) du_s(x) ds + \sigma W_t\,, \tag{1}$$

where U and Φ are vector fields and $\mathcal{L}(X_t) =: u_t$. It corresponds to a McKean-Vlasov diffusion which is a particular case of a nonlinear diffusion of the form

$$X_t = X_0 + \int_0^t U_1\left(X_s\right) ds + \int_0^t \int_{\mathbb{R}^d} U_2\left(X_s, x_2\right) du_s(x_2) ds + \sigma W_t\,. \tag{2}$$

Our work can be extended to general Diffusion (2). However, we will present our result in a particular case of McKean-Vlasov diffusion:

$$X_t = X_0 + \sigma W_t - \int_0^t \nabla V\left(X_s\right) ds - \alpha\zeta \int_0^t \left(X_s - \mathbb{E}[X_s]\right) ds\,, \tag{3}$$

V being a multi-wells landscape, α being positive and $\zeta \in \{-1; 1\}$. Here, $\Phi(x) = \pm\alpha x$ and $U(x) = -\nabla V(x)$.

Diffusion (3) corresponds to the probabilistic interpretation of a nonlinear partial differential equation, the granular media equation:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left\{\frac{\sigma^2}{2}\frac{\partial u}{\partial x} + \left(\nabla V + \alpha\zeta(x - m_t)\right) u\right\}\,, \tag{4}$$

where $m_t := \int_{\mathbb{R}^d} x du_t(x)$. The nonlinear diffusion also corresponds to the hydrodynamical limit of a mean-field system of particles:

$$X_t^i = X_0^i + \sigma W_t^i - \int_0^t \nabla V\left(X_s^i\right) ds - \alpha\zeta \int_0^t \left(X_s^i - \frac{1}{N}\sum_{j=1}^N X_s^j\right) ds\,. \tag{5}$$

Indeed, we can write (5) like so:

$$X_t^i = X_0^i + \sigma W_t^i - \int_0^t \nabla V\left(X_s^i\right) ds - \int_0^t \nabla F * \eta_s^N\left(X_s^i\right) ds\,, \tag{6}$$

where $\eta_s^N := \frac{1}{N}\sum_{j=1}^N \delta_{X_s^j}$ is the empirical measure and $F(x) := \alpha\zeta\frac{x^2}{2}$.

Intuitively, the particles $X^1, \cdots, X^N$ become independent so that η_t^N converges to $\mathcal{L}\left(X_t^1\right) = u_t$ as N goes to infinity. Consequently, as N becomes large, the drift in (6) becomes close to the one of (3).

1.3 The problem to solve

Here, we take $X_0 = x_0 \in \mathbb{R}^d$. We consider a domain $\mathcal{D} \subset \mathbb{R}^d$ which contains x_0 and we introduce

$$\tau_{\mathcal{D}}(\sigma) := \inf \{t \geq 0 \ : \ X_t \notin \mathcal{D}\}$$

the first exit-time of X from the domain $\mathcal{D}$. The exit-problem consists of two questions. What is the exit-time ? What is the exit-location ? The subject of this article is to study the first one. Indeed, we can solve the exit-location question like in [Tug12], by using the results on the exit-time. In the small-noise limit, the question becomes: “What is the exit-time $\tau_{\mathcal{D}}(\sigma)$ for σ going to 0 ?”

More precisely, we aim to establish a Kramers'type law:

$$\lim_{\sigma \to 0} \mathbb{P} \left\{ \exp \left[\frac{2}{\sigma^2}(H - \delta) \right] \leq \tau_{\mathcal{D}}(\sigma) \leq \exp \left[\frac{2}{\sigma^2}(H + \delta) \right] \right\} = 1 \,,$$

for any $\delta > 0$. Here, H corresponds to the exit cost. Let us remark that this exit cost has been computed in [HT14].

1.4 Freidlin-Wentzell theory

The natural framework is the one of the large deviations. Freidlin and Wentzell theory solves the exit-problem for the time-homogeneous diffusions. Let us briefly present this theory. We refer the reader to [FW98, DZ98] for a complete review. We look at the diffusion

$$x_t^\sigma = x_0 + \sigma \beta_t - \int_0^t \nabla U \left(x_s^\sigma \right) ds \,.$$

U is a $\mathcal{C}^\infty$-continuous function from $\mathbb{R}^k$ ($k \geq 1$) to $\mathbb{R}$ and β is a Brownian motion in $\mathbb{R}^k$. Let a_0 be a minimizer of U and $\mathcal{G}$ be a domain which contains a_0.

We also consider the deterministic path $\Psi(x_0)$:

$$\Psi_t(x_0) = x_0 - \int_0^t \nabla U \left(\Psi_s(x_0) \right) \,.$$

Then:

$$\lim_{\sigma \to 0} \mathbb{P} \left\{ \sup_{t \in [0;T]} ||x_t^\sigma - \Psi_t(x_0)|| > \delta \right\} = 0 \,,$$

for any $T, \delta > 0$.

Moreover, under easily checked assumptions, for any $\delta > 0$, the following Kramers' type law holds:

$$\lim_{\sigma \to 0} \mathbb{P} \left\{ \exp \left[\frac{2}{\sigma^2} \left(H - \delta \right) \right] < \tau_{\mathcal{G}}(\sigma) < \exp \left[\frac{2}{\sigma^2} \left(H + \delta \right) \right] \right\} = 1 \,.$$

Here, the exit cost is $H := \inf_{z \in \partial \mathcal{G}} \left[U(z) - U(a_0) \right]$. We immediately remark that

$$H = \lim_{\sigma \to 0} \frac{\sigma^2}{2} \log \left\{ \mathbb{E} \left[\tau_{\mathcal{G}}(\sigma) \right] \right\} \,.$$

1.5 Previous results on the subject

Let us present four methods which have been investigated to solve the exit-problem for McKean-Vlasov diffusion.

1.5.1 The method in [HIP08]

Herrmann, Imkeller and Peithmann have solved the problem with a convex confining potential. They reconstruct the theory of Freidlin-Wentzell to the inhomogeneous Diffusion (1). But, their method relies completely on the convexity of the confining potential so that it can not be extended to the nonconvex case.

1.5.2 The method initiated in [Tug10]

The problem in the previous method is the convergence in long time. An idea thus is to study the long-time convergence of the drift $\nabla V + \nabla F * u_t$, that we have solved in [Tug13a, Tug13b]. If we are able to find the rate of convergence (that has been made in [BGG13] albeit in the convex case), we are able to solve the exit-problem. However, to find this rate is a difficult question. Moreover, this rate of convergence is strongly linked to the exit-time in homogeneous diffusion so that we expect that it also is linked in the McKean-Vlasov diffusion.

1.5.3 The method in [Tug14]

The idea in a previous paper is to make a coupling between Diffusion (3) and another McKean-Vlasov diffusion:

$$Y_t = x_0 + \sigma W_t - \int_0^t \nabla V_0\left(Y_s\right) ds - \alpha\zeta \int_0^t \left(Y_s - \mathbb{E}[Y_s]\right) ds\,, \tag{7}$$

where V_0 is uniformly strictly convex and is equal to V except on a compact $\mathcal{K}$. We thus consider a domain $\mathcal{D} \subset \mathcal{K}^c$.

The difficulty is to find a good upper-bound for the first hitting-time of the compact $\mathcal{K}$.

Under a technical assumption, we are able to obtain a Kramers'type law for the domain $\mathcal{D}$. However, this assumption does not seem fundamental.

1.5.4 The method in [Tug12]

In this previous article, we proceed differently. We have solved the exit-problem of the first particle in the mean-field system of particles as noise elapses and the number of particles is large. Then, a coupling result makes the Kramers'type law for the first particle to hold for the McKean-Vlasov diffusion. However, to ensure that the law $\mathcal{L}\left(X_t\right)$ is close to δ_a (a being the point in which the minimum of V is reached), we need the convexity of the confining potential.

1.6 Method of this article

We make a coupling between the McKean-Vlasov diffusion (3) and the time-homogeneous diffusion:

$$Z_t = x_0 + \sigma W_t - \int_0^t \nabla V\left(Z_s\right) ds - \alpha\zeta \int_0^t \left(Z_s - a\right) ds\,, \tag{8}$$

a being the unique wells of the potential V on the domain $\mathcal{D}$.

We need to control the law $\mathcal{L}\left(X_t\right)$. We then introduce, for small $\kappa > 0$, two deterministic times $T^i_\kappa(\sigma)$ and $T^s_\kappa(\sigma)$. The time $T^i_\kappa(\sigma)$ corresponds to the first time such that the Wasserstein distance between δ_a and $\mathcal{L}\left(X_t\right)$ is less than κ. And, the time $T^s_\kappa(\sigma)$ is such that the probability of some exit-time to be less than $T^s_\kappa(\sigma)$ is equal to κ.

Then, we establish a link between $T^s_\kappa(\sigma)$ and $\mathcal{L}\left(X_t\right)$.

We thus obtain $\mathbb{P}\left(\mathcal{T}_0 \leq \exp\left[\frac{2}{\sigma^2}\left(H - \delta\right)\right]\right) \longrightarrow 0$ as σ goes to 0. Here, $\mathcal{T}_0$ is an exit-time and H is an exit cost. From this limit, we are able to apply the results in [Tug14].

The idea is that the diffusion Z is close to X and X^1. However, we know the exit cost, see [Tug15a, HT14].

1.7 Outline of the paper

First, we give the assumptions and the notations. Then, we give the main results, that is to say Kramers'type law for McKean-Vlasov diffusion evolving in a non-convex landscape. The following sections deal with the proofs of the theorems. Section 4 gives the proof of Theorem A. Section 5 gives the proof of Proposition B. Finally, Section 6 gives the proof of Theorem C.

2 Assumptions and notations

In this work, we take less general hypotheses than the ones in [Tug15b].

Assumption 2.1. *The potentials V and F satisfy the following hypotheses:*

- *The coefficients ∇V and ∇F are locally Lipschitz, that is, for each $R > 0$ there exists $K_R > 0$ such that*

$$||\nabla V(x) - \nabla V(y)|| + ||\nabla F(x) - \nabla F(y)|| \leq K_R\,||x - y||\,,$$

  for $x, y \in \left\{z \in \mathbb{R}^d \;:\; ||z|| < R\right\}$.

- *The function V is continuously differentiable.*
- *The potential V is convex at infinity: $\lim\limits_{||x||\to+\infty} \nabla^2 V(x) = +\infty$.*
- *$F(x) := \alpha\zeta\frac{||x||^2}{2}$, $\alpha > 0$ and $\zeta \in \{-1; 1\}$.*

Since the initial law is a Dirac measure, we know that there exists a unique strong solution X to Equation (3), see Theorem 2.13 in [HIP08] for a proof. Moreover:

$$\sup_{t\in\mathbb{R}_+}\mathbb{E}\left\{||X_t||^{2p}\right\}<\infty \tag{9}$$

for all $p\in\mathbb{N}^*$.

We now give an essential definition which is of crucial interest in large deviations.

Definition 2.2. *Let k be any positive integer. Let $\mathcal{G}$ be a subset of $\mathbb{R}^k$ and let U be a vector field from $\mathbb{R}^k$ to $\mathbb{R}^k$ which satisfies the set of assumptions 2.1. For all $x\in\mathbb{R}^k$, we consider the dynamical system*

$$\psi_t(x)=x+\int_0^t U\left(\psi_s(x)\right)ds\,.$$

We say that the domain $\mathcal{G}$ is stable by U if the orbit $\{\psi_t(x)\,;\,t\in\mathbb{R}_+\}$ is included in $\mathcal{G}$ for all $x\in\mathcal{G}$.

Assumption 2.3. *We consider the dynamical system*

$$\varphi_t=x_0-\int_0^t\nabla V\left(\varphi_s\right)ds\,.$$

The orbit $\{\varphi_t\ ;\ t\geq 0\}$ is included into the domain $\mathcal{D}$. Moreover, $\varphi_\infty=a\in\mathcal{D}$.

This hypothesis is natural. Indeed, since the dynamical system and the diffusion are close when the noise is small, if the Assumption 2.3 is not satisfied, there exists $T>0$ independent from the noise such that the probability for the diffusion to exit from $\mathcal{D}$ before T is close to 1.

Assumption 2.4. *The open domain $\mathcal{D}$ is stable by the vector field*

$$x\mapsto-\nabla V(x)-\nabla F(x-a)\,.$$

Moreover, $\nabla^2V(x)+\alpha\zeta$ is uniformly positive on $\mathcal{D}$.

The stability of $\mathcal{D}$ for the vector field $x\mapsto-\nabla V(x)-\nabla F(x-a)$ permits us to solve the exit-problem of Diffusion (8) from $\mathcal{D}$. The positivity of the quantity $\inf_{x\in\mathcal{D}}\nabla^2V(x)+\alpha\zeta$ ensures the convexity of the potential $x\mapsto V(x)+F(x-a)$, that is necessary for having a good coupling between Difusions (3) and (8).

Assumption 2.5. *If $\zeta=+1$, there exists $\rho>0$ such that for all $x\in\mathcal{D}$, we have*

$$\langle x-a\,;\,\nabla V(x)\rangle\geq\rho\,||x-a||^2\,.$$

If $\zeta=-1$, there exists $\rho>0$ such that for all $x\in\mathcal{D}$, we have

$$\langle x-a\,;\,\nabla V(x)-\alpha(x-a)\rangle\geq\rho\,||x-a||^2\,.$$

The Hypothesis 2.5 is technical but is necessary to control the moments of $\mathcal{L}(X_t)$.

Assumption 2.6. *If $\zeta = +1$, the potential V is uniformly strictly convex on $\mathcal{D}$. If $\zeta = -1$, the potential $x \mapsto V(x) + F(x-a)$ is uniformly strictly convex on $\mathcal{D}$.*

Let us remark that Assumption 2.4 implies Assumption 2.6 if $\zeta = -1$.

Assumption 2.6 is necessary to apply the results in [Tug14].

3 Main results

Definition 3.1. *We define the exit-time:*

$$\tau_{\mathcal{D}}(\sigma) := \inf\left\{t > 0 \;:\; X_t \notin \mathcal{D}\right\}$$

and its associated exit-cost:

$$H := \inf_{x\in\partial\mathcal{D}} \left(V(x) + F(x-a) - V(a)\right)\,.$$

Theorem A: *Under Hypotheses 2.1–2.5, we have the limit:*

$$\lim_{\sigma\to 0} \mathbb{P}\left\{\tau_{\mathcal{D}}(\sigma) \leq \exp\left[\frac{2}{\sigma^2}\left(H-\delta\right)\right]\right\} = 0\,, \tag{10}$$

for any $\delta > 0$.

Next result does not provide the Kramers'type law but is a first step in this direction. However, we can not extend it by the method developed in this paper.

Proposition B: *Under Hypotheses 2.1–2.5, we have the limit:*

$$\liminf_{\sigma\to 0} \mathbb{P}\left\{\tau_{\mathcal{D}}(\sigma) \leq \exp\left[\frac{2}{\sigma^2}\left(H+\delta\right)\right]\right\} > 0\,, \tag{11}$$

for any $\delta > 0$.

Theorem C: *Under Hypotheses 2.1–2.6, we have the limit:*

$$\lim_{\sigma\to 0} \mathbb{P}\left\{\tau_{\mathcal{D}}(\sigma) \geq \exp\left[\frac{2}{\sigma^2}\left(H+\delta\right)\right]\right\} = 0\,, \tag{12}$$

for any $\delta > 0$.

Theorem C together with Theorem A immediately gives the following result:

Corollary D: *Under Hypotheses 2.1–2.6, we have the Kramers'law:*

$$\lim_{\sigma\to 0} \mathbb{P}\left\{\exp\left[\frac{2}{\sigma^2}\left(H-\delta\right)\right] \leq \tau_{\mathcal{D}}(\sigma) \leq \exp\left[\frac{2}{\sigma^2}\left(H+\delta\right)\right]\right\} = 1\,, \tag{13}$$

for any $\delta > 0$. In particular, we have the limit:

$$\lim_{\sigma\to 0} \frac{\sigma^2}{2} \log\left\{\mathbb{E}\left[\tau_{\mathcal{D}}(\sigma)\right]\right\} = H$$

We do not provide result on the exit location since we can solve it like it has been done in [Tug12].

4 Proof of Theorem A

4.1 Control of the moments

Lemma 4.1. *Under the Assumptions 2.1-2.5, we have:*

$$\frac{d}{dt}\mathbb{E}\left[||X_t - a||^2\right] \leq -2\rho\mathbb{E}\left[||X_t - a||^2\right] + K\sqrt{\mathbb{P}\left(X_t \notin \mathcal{D}\right)} + \sigma^2\,,$$

K being a positive constant.

Proof. By Itô formula, we have:

$$\begin{aligned} d\,||X_t - a||^2 = 2\sigma \left\langle\, X_t - a;\, dW_t\right\rangle - 2\left\langle\, X_t - a;\, \nabla V(X_t)\right\rangle dt \\ - 2\alpha\zeta\left\langle\, X_t - a;\, X_t - \mathbb{E}[X_t]\right\rangle dt + \sigma^2 dt\,. \end{aligned}$$

We assume $\zeta = +1$. Then,

$$\mathbb{E}\left\{\left\langle\, X_t - a;\, X_t - \mathbb{E}[X_t]\right\rangle\right\} \geq 0\,.$$

We integrate, we take the expectation then we take the derivative. We thus obtain:

$$\frac{d}{dt}\mathbb{E}\left[||X_t - a||^2\right] \leq -2\mathbb{E}\left[\left\langle\, X_t - a;\, \nabla V(X_t)\right\rangle\right] + \sigma^2\,.$$

However,

$$\left\langle\, X_t - a;\, \nabla V(X_t)\right\rangle = \left\langle\, X_t - a;\, \nabla V(X_t)\right\rangle \mathbb{1}_{X_t \in \mathcal{D}} + \left\langle\, X_t - a;\, \nabla V(X_t)\right\rangle \mathbb{1}_{X_t \notin \mathcal{D}}\,.$$

Consequently, we have:

$$\begin{aligned} \frac{d}{dt}\mathbb{E}\left[||X_t - a||^2\right] \leq &- 2\rho\mathbb{E}\left[||X_t - a||^2\right] + \sigma^2 \\ &+ 2\mathbb{E}\left\{\left[\rho\,||X_t - a||^2 - \left\langle\, X_t - a;\, \nabla V(X_t)\right\rangle\right]\mathbb{1}_{X_t \notin \mathcal{D}}\right\} \end{aligned}$$

The uniform boundedness of the moments and Cauchy-Schwarz inequality yield the existence of a positive constant K such that

$$\frac{d}{dt}\mathbb{E}\left[||X_t - a||^2\right] \leq -2\rho\mathbb{E}\left[||X_t - a||^2\right] + K\sqrt{\mathbb{P}\left(X_t \notin \mathcal{D}\right)} + \sigma^2\,,$$

which achieves the proof.

The proof if we assume $\zeta = -1$ is similar. □

Let us note that we could have used Hölder inequality in order to get a better bound. But, in this work, to have the square root is sufficient.

Remark 4.2. *For any $T > 0$, we have $\mathbb{P}\left\{\tau_{\mathcal{D}}(\sigma) \leq T)\right\} \longrightarrow 0$ as σ goes to 0.*

This is a classical result from large deviations.

Lemma 4.3. *For any $\kappa > 0$, there exist $\sigma_0 > 0$, $T_\kappa(\sigma_0) > 0$ such that for any $\sigma < \sigma_0$, there exists $T^i_\kappa(\sigma) \leq T_\kappa(\sigma_0)$ satisfying*

$$\mathbb{E}\left[\left|\left|X_{T^i_\kappa(\sigma)} - a\right|\right|^2\right] \leq \kappa \,.$$

Proof. It is a straightforward consequence of previous lemma. □

Definition 4.4. *By $T^s_\kappa(\sigma)$, we denote the unique positive real which satisfies*

$$\mathbb{P}\left(\tau_{\mathcal{D}}(\sigma) \leq T^s_\kappa(\sigma)\right) = \kappa \,.$$

By classical result on large deviations, we know $\lim\limits_{\sigma\to 0} T^s_\kappa(\sigma) = +\infty$.

4.2 Probability of exiting before $T^i_\kappa(\sigma)$

Due to the boundedness of $T^i_\kappa(\sigma)$, classical results on Freidlin-Wentzell theory implies:

$$\mathbb{P}\left(\tau_{\mathcal{D}}(\sigma) \leq T^i_\kappa(\sigma)\right) \longrightarrow 0$$

as σ goes to 0, under Assumption 2.3.

4.3 Coupling result

Definition 4.5. *We consider the diffusion $Y := (Y_t)_{t \geq T^i_\kappa(\sigma)}$ defined by*

$$\begin{aligned} Y_{T^i_\kappa(\sigma)+t} = X_{T^i_\kappa(\sigma)} + \sigma\left(W_{T^i_\kappa(\sigma)+t} - W_{T^i_\kappa(\sigma)}\right) - \int_{T^i_\kappa(\sigma)}^{T^i_\kappa(\sigma)+t} \nabla V\left(Y_s\right) ds \\ - \alpha\zeta \int_{T^i_\kappa(\sigma)}^{T^i_\kappa(\sigma)+t} \left(Y_s - a\right) ds \,. \end{aligned}$$

Lemma 4.6. *For any $\xi > 0$, under the Assumptions 2.1-2.4, we have the equality:*

$$\mathbb{P}\left\{\sup_{t\in[T^i_\kappa(\sigma);T^s_\kappa(\sigma)]} ||X_t - Y_t|| \geq \xi\right\} = 0$$

if κ and σ are small enough.

Proof. If σ is small enough, we have $T^i_\kappa(\sigma) < T^s_\kappa(\sigma)$. Differential calculus provides

$$d\,||X_t - Y_t||^2 = -2\,\langle\, X_t - Y_t;\, \nabla W_{\mu_t}(X_t) - \nabla W_{\delta_a}(Y_t)\rangle\, dt\,,$$

where $W_\mu(x) := V(x) + F * \mu(x)$ and $\mu_t := \mathcal{L}\left(X_t\right)$.

For any $T^i_\kappa(\sigma) \leq t \leq T^s_\kappa(\sigma)$, we have:

$$\begin{aligned} d\,||X_t - Y_t||^2 =& -2\,\langle\, X_t - Y_t;\, \nabla W_{\delta_a}(X_t) - \nabla W_{\delta_a}(Y_t)\rangle\, dt \\ & -2\,\langle\, X_t - Y_t;\, \nabla W_{\mu_t}(X_t) - \nabla W_{\delta_a}(X_t)\rangle\, dt \\ \leq & -2\theta\,||X_t - Y_t||^2 + 2\sqrt{\kappa}\,||X_t - Y_t||\,. \end{aligned}$$

However, $X_{T^i_\kappa(\sigma)} = Y_{T^i_\kappa(\sigma)}$. Thus, for any $t \in \left[T^i_\kappa(\sigma); T^s_\kappa(\sigma)\right]$, we have:

$$||X_t - Y_t|| \leq \frac{\sqrt{\kappa}}{\theta}\,.$$

By taking $\kappa < \theta^2\xi^2$, we have the result. $\square$

4.4 Proof of the inequality (10)

We proceed a *reducto ad absurdum* by assuming $T^s_\kappa(\sigma) \leq \exp\left[\frac{2}{\sigma^2}(H-\delta)\right]$, for some $\delta > 0$ and for any $\kappa > 0$. We know by definition that

$$\mathbb{P}\left(\tau_{\mathcal{D}}(\sigma) \leq T^s_\kappa(\sigma)\right) = \kappa\,.$$

However, we have:

$$\mathbb{P}\left(\tau_{\mathcal{D}}(\sigma) \leq T^s_\kappa(\sigma)\right) = \mathbb{P}\left(\tau_{\mathcal{D}}(\sigma) \leq T^i_\kappa(\sigma)\right) + \mathbb{P}\left(T^i_\kappa(\sigma) \leq \tau_{\mathcal{D}}(\sigma) \leq T^s_\kappa(\sigma)\right)\,.$$

The first term in the right-hand side goes to 0 as σ goes to 0.

Let us prove that the second term also goes to 0 as the noise elapses.

By $\tau'(\sigma)$, we denote the first exit-time of diffusion Y from a domain $\mathcal{D}'$ such that

1. $\mathcal{D}'$ is stable by the vector field $x \mapsto -\nabla V(x) - \nabla F(x-a)$,
2. $\mathcal{D}' \subset \mathcal{D}$,
3. dist $(\mathcal{D}'; \mathcal{D}^c) =: \xi > 0$,
4. the exit cost of diffusion Y from $\mathcal{D}'$ is larger than $H - \frac{\delta}{2}$.

The existence of such a domain is a straightforward exercice so it is left to the reader. We have:

$$\begin{aligned}\mathbb{P}\left(T^i_\kappa(\sigma) \leq \tau_{\mathcal{D}}(\sigma) \leq T^s_\kappa(\sigma)\right) \leq& \mathbb{P}\left(T^i_\kappa(\sigma) \leq \tau'(\sigma) \leq T^s_\kappa(\sigma)\right)\\ &+ \mathbb{P}\left(\sup_{[T^i_\kappa(\sigma);T^s_\kappa(\sigma)]} ||X_t - Y_t|| \geq \xi\right)\,.\end{aligned}$$

By taking κ and σ small enough, the second term is equal to 0. Then, we observe the following inequality

$$\mathbb{P}\left(T^i_\kappa(\sigma) \leq \tau'(\sigma) \leq T^s_\kappa(\sigma)\right) \leq \mathbb{P}\left(\tau'(\sigma) \leq \exp\left[\frac{2}{\sigma^2}(H-\delta)\right]\right)\,.$$

However, this quantity goes to 0 since the exit cost of Y from domain $\mathcal{D}'$ is larger than $H - \frac{\delta}{2}$. We obtain $\kappa = 0$ which is absurd.

We deduce that for any $\kappa > 0$ small enough, there exists $\sigma_\kappa > 0$ such that for any $0 < \sigma < \sigma_\kappa$, we have

$$\exp\left[\frac{2}{\sigma^2}(H-\delta)\right] \leq T^s_\kappa(\sigma).$$

Immediately, we obtain

$$\mathbb{P}\left(\tau_{\mathcal{D}}(\sigma)\leq\exp\left[\frac{2}{\sigma^2}\left(H-\delta\right)\right]\right)<\kappa$$

for any $\kappa>0$ if σ is small enough. Consequently, we obtain:

$$\lim_{\sigma\to0}\mathbb{P}\left(\tau_{\mathcal{D}}(\sigma)\leq\exp\left[\frac{2}{\sigma^2}\left(H-\delta\right)\right]\right)=0\,.$$

5 Proof of Proposition B

Let us now prove Inequality (11). We proceed similarly by assuming that $T_\kappa^s(\sigma)\geq\exp\left[\frac{2}{\sigma^2}\left(H+\delta\right)\right]$. Then, we have:

$$\mathbb{P}\left(\tau_{\mathcal{D}}(\sigma)\leq\exp\left[\frac{2}{\sigma^2}\left(H+\delta\right)\right]\right)\leq\kappa\,.$$

However, we have:

$$\mathbb{P}\left(\tau_{\mathcal{D}}(\sigma)\leq\exp\left[\frac{2}{\sigma^2}\left(H+\delta\right)\right]\right)=\mathbb{P}\left(\tau_{\mathcal{D}}(\sigma)\leq T_\kappa^i(\sigma)\right)+\mathbb{P}\left(T_\kappa^i(\sigma)\leq\tau_{\mathcal{D}}(\sigma)\leq\exp\left[\frac{2}{\sigma^2}\left(H+\delta\right)\right]\right)\,.$$

The first term goes to 0 as σ goes to 0. Let us now prove that the second term converges to 1 as the noise elapses. Indeed, we have

$$\begin{aligned}&\mathbb{P}\left(T_\kappa^i(\sigma)\leq\tau_{\mathcal{D}}(\sigma)\leq\exp\left[\frac{2}{\sigma^2}\left(H+\delta\right)\right]\right)\\&\geq\mathbb{P}\left(T_\kappa^i(\sigma)\leq\tau'(\sigma)\leq\exp\left[\frac{2}{\sigma^2}\left(H+\delta\right)\right]\right)\\&-\mathbb{P}\left(\sup_{t\in[T_\kappa^i(\sigma);T_\kappa^s(\sigma)]}||X_t-Y_t||\geq\xi\right)\,,\end{aligned}$$

where $\tau'(\sigma)$ is the exit-time of Y from a domain $\mathcal{D}'$ such that

1. $\mathcal{D}'$ is stable by the vector field $x\mapsto-\nabla V(x)-\nabla F(x-a)$,
2. $\mathcal{D}\subset\mathcal{D}'$,
3. dist $\left(\mathcal{D};(\mathcal{D}')^c\right)=:\xi>0$,
4. the exit cost of diffusion Y from $\mathcal{D}'$ is less than $H+\frac{\delta}{2}$.

The second term is equal to 0 if κ and σ are small enough. The first term goes to 1 as σ goes to 0.

We deduce that $T_\kappa^s(\sigma)<\exp\left[\frac{2}{\sigma^2}\left(H+\delta\right)\right]$ if σ and κ are small enough.

This achieves the proof.

6 Proof of Theorem C

We proceed like in [Tug14] so we skip the details.

We introduce $\mathcal{K}$ a compact which contains (in its interior) the compact in which V is not convex. Thus, there exists a convex potential V_0 such that $V - V_0 = (V - V_0)\,\mathbb{1}_{\mathcal{K}}$.

We consider the diffusion

$$Z_t = X_0 + \sigma W_t - \int_0^t \nabla V_0\,(Z_s)\,ds - \int_0^t \nabla F * \nu_s\,(Z_s)\,ds\,,$$

where $\nu_t := \mathcal{L}\,(Z_t)$.

We rewrite Lemma 3.1 in [Tug14]:

Lemma 6.1. *For any positive t, we have*

$$\mathbb{E}\left\{||X_t - Y_t||^2\right\} \leq \frac{M^2}{\theta^2}\mathbb{P}\,(\mathcal{T}_0 \leq t)\ ,$$

where $\mathcal{T}_0$ is the first hitting time of diffusion X of the compact set $\mathcal{K}$ and $M := \sup\limits_{\mathcal{K}} ||\nabla V - \nabla V_0||$.

The proof is omitted because it has been made in [Tug14] under assumption 2.6 if $\zeta = +1$. The proof if $\zeta = -1$ is similar.

We now give Lemma 3.2 in [Tug14]:

Lemma 6.2. *Whenever Λ is positive, for any t positive, we have*

$$\mathbb{E}\left\{\sup_{t\leq\mathcal{T}_0} ||X_t - Y_t||^2\,\mathbb{1}_{\mathcal{T}_0 \leq e^{\frac{2\Lambda}{\sigma^2}}}\right\} \leq K\mathbb{P}\left(\mathcal{T}_0 \leq \exp\left[\frac{2}{\sigma^2}\Lambda\right]\right)\ ,$$

K being a positive constant.

Proposition 3.3 in [Tug14] gives result about Λ such that

$$\mathbb{P}\left(\mathcal{T}_0 \leq \exp\left[\frac{2}{\sigma^2}\Lambda\right]\right)$$

goes to 0 as σ goes to 0. But, in this paper, we have obtained a better result.

Thanks to Theorem A, we have

$$\lim_{\sigma\to 0} \mathbb{P}\left(\mathcal{T}_0 \leq \exp\left[\frac{2}{\sigma^2}\,(H_0 - \delta)\right]\right) = 0\,,$$

for any $\delta > 0$. Here, H_0 is the exit cost of $\mathcal{D}_0$, the maximal domain satisfying assumptions 2.3–2.6.

We thus apply Theorem 4.2 in [Tug14] and we obtain

$$\lim_{\sigma\to 0} \mathbb{P}\left(\tau_{\mathcal{D}}(\sigma) \geq \exp\left[\frac{2}{\sigma^2}\,(H + \delta)\right]\right) = 0\,.$$

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